[0.1] Sets

In mathematics, we do not define the word *"set",* but assume an understanding of it. Unlike most terms used in mathematics, the word "set" has the same sense it has in ordinary language where it means a collection of objects. All we require is that a set be specified in such a way that any particular object either does or does not belong to it.

■ Notation used with sets

Typically we name a set by using a capital letter. We describe a set either by listing the objects that belong to it or by stating a condition that an object must meet in order to be included in it. We use braces { } to show that a description of a set is enclosed.

■ Specifying a set by listing its members

- $\mathcal{A} = \{1, 2, 3\}$ tells us that $\mathcal{A}$ names the set consisting of items 1, 2, 3.

Often we may indicate a set to which many objects belong by listing enough of the objects so that it is obvious which objects are included in the set and using "…" to show that some objects have not been written.

- $\mathcal{A} = \{1, 2, 3, 4, \ldots, 100\}$ tells us that $\mathcal{A}$ names the set of natural numbers from *1* to *100*.
- $\mathcal{A} = \{1, 2, 3, 4, \ldots\}$ tells us that $\mathcal{A}$ names the set of all natural numbers.

■ Specifying a set by stating a condition

As mentioned, we may indicate a set by stating a condition that an object must meet to get into the set.

- $\mathcal{A}$ = {natural numbers less than 5} tells us which objects belong to the set $\mathcal{A}$. Since *1, 2, 3, 4* satisfy the condition, they belong to set $\mathcal{A}$. But *5* does not meet the condition, therefore it does not belong to set $\mathcal{A}$.

We write "$\mathcal{A}$ = {natural numbers less that 5}" as follows:

- $\mathcal{A} = \{x : x$ is a natural number *and* x is less than $5\}$

We read the sign "**:**" as "such that", so that in English this example is read "the set of all x such that x is a natural number *and* x is less than *5* ". The notation used in this example is called **set builder notation**.

■ Element (or member) of a set.

When an object belongs to a particular set, we say that the object is an **element** or a **member** of that set. For example, *2* is an element of the set {2, 3, 4, 7}. The symbol $\in$ means "is an element of " and the symbol $\notin$ means "is not an element of ".

- $2 \in \{1, 2, 5, 10\}$ says that *2* is an element of the set {1, 2, 5, 10}.
- $2 \notin \{1, 3, 5, 10\}$ says that *2* is not an element of the set {1, 4, 8, 12}.

■ Equality of sets

Two sets are equal *if and only if* they contain exactly the same elements.

- If $\mathcal{A} = \{2, 4, 6, 8\}$ and $\mathcal{B} = \{x : x \text{ is an even number less than } 10\}$, then $\mathcal{A} = \mathcal{B}$.

■ Set identity

The identity of a set is unchanged by either repetition of its elements or the arrangement of its elements.

- $\{1, 2, 3, 4, 5, 6\} = \{3, 2, 2, 2, 1, 4, 6, 5, 5\}$
- $\{1, 2, 3\} = \{2, 1, 3\} = \{3, 2, 1\}$

■ Subset

When every element of a set $\mathcal{A}$ is an element of a set $\mathcal{B}$, we say that $\mathcal{A}$ is a **subset** of $\mathcal{B}$. Symbolically, $\mathcal{A} \subseteq \mathcal{B}$ says "$\mathcal{A}$ is a subset of $\mathcal{B}$ ". To say that $\mathcal{A}$ is not a subset of $\mathcal{B}$, we write $\mathcal{A} \nsubseteq \mathcal{B}$.

- If $\mathcal{A} = \{7, 11, 12, 21\}$, $\mathcal{B} = \{7, 12\}$, $C = \{7, 15\}$, then $\mathcal{B} \subseteq \mathcal{A}$ and $C \nsubseteq \mathcal{A}$.

A formal definition of subset is:

◆ Def. $\mathcal{A} \subseteq \mathcal{B}$ if and only if $x \in \mathcal{A}$ implies $x \in \mathcal{B}$.

- $\{1, 2, 4\} \subseteq \{1, 2, 3, 4, 5\}$

Note that this definition allows that $\mathcal{A} \subseteq \mathcal{A}$, since $a \in \mathcal{A}$ implies $a \in \mathcal{A}$. That is, every set is a subset of itself.

- $\{2, 5, 6, 9\} \subseteq \{2, 5, 6, 9\}$

- **Proper subset**

If $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A}$ is a **proper subset** of $\mathcal{B}$; written $\mathcal{A} \subset \mathcal{B}$.

- $\{2, 5, 6\} \subset \{2, 5, 6, 9\}$

> Be careful using the signs $\in$ and $\subseteq$. If $\mathcal{A} = \{1, 2, 3, 4, 5, 6\}$, then $2 \in \mathcal{A}$ and $\{2, 3\} \subseteq \mathcal{A}$ and $\{2\} \subseteq \mathcal{A}$. It is false that $\{2\} \in \mathcal{A}$. However, $\{2\} \in \{3, 4, 5, \{2\}, 7\}$ is true.

Using the idea of subset, we give a definition that will be useful in working proofs.

◆ **Def.** $\mathcal{A} = \mathcal{B}$ **if and only if** $\mathcal{A} \subseteq \mathcal{B}$ **and** $\mathcal{B} \subseteq \mathcal{A}$**.**

- **The empty set**

We call the set that has no members **the empty set**. We indicate the empty set either by { } or by $\varnothing$. The empty set is a subset of every set; that is, for any set $\mathcal{A}$, $\varnothing \subseteq \mathcal{A}$.

- **The universal set**

The universal set, indicated by $\mathcal{U}$, is the set that contains all the objects being considered in a given discussion.

Operations with sets

- **Complement of a set**

The **complement** of a set $\mathcal{A}$ is the set of all elements of the universal set that are not elements of $\mathcal{A}$. We indicate the complement of $\mathcal{A}$ by the symbol $\overline{\mathcal{A}}$.

- If $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{A} = \{5, 6, 7\}$, then $\overline{\mathcal{A}} = \{1, 2, 3, 4\}$.

Intersection

Given sets $\mathcal{A}$ and $\mathcal{B}$, the **intersection** of $\mathcal{A}$ and $\mathcal{B}$ is the set of all objects that are in both $\mathcal{A}$ ***and*** $\mathcal{B}$. $\mathcal{A} \cap \mathcal{B}$ is read "$\mathcal{A}$ intersect $\mathcal{B}$". Formally,

◆ **Def.** $\mathcal{A} \cap \mathcal{B} = \{x : x \in \mathcal{A} \text{ and } x \in \mathcal{B}\}$

- If $\mathcal{A} = \{0, 2, 4, 6, 8, 100\}$ and $\mathcal{B} = \{0, 1, 3, 5, 7, 9, 100\}$, then $\mathcal{A} \cap \mathcal{B} = \{0, 100\}$

Union

Given sets $\mathcal{A}$ and $\mathcal{B}$, the **union** of $\mathcal{A}$ and $\mathcal{B}$ is the set of all objects that are in either $\mathcal{A}$ ***or*** $\mathcal{B}$ or **both** $\mathcal{A}$ **or** $\mathcal{B}$. $\mathcal{A} \cup \mathcal{B}$ is read "$\mathcal{A}$ union $\mathcal{B}$". Formally,

◆ **Def.** $\mathcal{A} \cup \mathcal{B} = \{x : x \in \mathcal{A} \text{ or } x \in \mathcal{B}\}$

- If $\mathcal{A} = \{0, 1, 2, 3\}$ and $\mathcal{B} = \{0, 4, 5, 6, 7\}$, then $\mathcal{A} \cup \mathcal{B} = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Further qualities of sets

Cardinality of a set

The **cardinality** of a set is the number of elements in the set. When we wish to consider the cardinality of a set $\mathcal{A}$, we write $n\mathcal{A}$.

- The cardinality of $\mathcal{A} = \{1, 10, 100, 1000, 10\,000\}$ is 5, written $n\mathcal{A} = 5$.

- The cardinality of $\{a, b, c, d, e, f\}$ is 6.

One-to-one correspondence

Consider sets $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{B} = \{a, b, c\}$. We can match each element of $\mathcal{A}$ with each element of $\mathcal{B}$ using every element *once and only once*. For example,

$$\begin{array}{ccc} 1 & 2 & 3 \\ \updownarrow & \updownarrow & \updownarrow \\ b & c & a \end{array}$$

When such a matching can be made, we say the sets have a **one-to-one correspondence**.

Equivalence

If sets $\mathcal{A}$ and $\mathcal{B}$ have a one-to-one correspondence, we say that sets $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**.

Note that two sets that are equivalent need not be equal. For example,

- Suppose $\mathcal{A} = \{a, b, d\}$ and $\mathcal{B} = \{5, 21, 3\}$. Clearly, $\mathcal{A} \neq \mathcal{B}$. However, $\mathcal{A}$ is equivalent to $\mathcal{B}$, since

$$\begin{array}{ccc} a & b & d \\ \updownarrow & \updownarrow & \updownarrow \\ 5 & 21 & 3 \end{array}$$

exhibits a one-to-one correspondence of $\mathcal{A}$ and $\mathcal{B}$.

■ Some important sets with which you are already familiar

$\mathcal{N} = \{1, 2, 3, \ldots\}$	The set of natural numbers.
$\mathcal{W} = \{0, 1, 2, 3, \ldots\}$	The set of whole numbers.
$\mathcal{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	The set of integers.
$\mathcal{Z}^+ = \{1, 2, 3, \ldots\}$	The set of positive integers.
$\mathcal{Z}^- = \{-1, -2, -3, \ldots\}$	The set of negative integers.

The letter $\mathcal{Z}$ was chosen, because "*Zahlen* " means number in German. The term "positive integer" is used much more often than is the term "natural number".

■ Some important sets with which you will become familiar

$\mathbb{R}$ The set of real numbers. The set of all numbers that can be represented by points on a line.

The set of irrational numbers is the set of all real numbers that are not rational.

Questions on Sets

List the elements of each set

1. The set of positive integers between *3* and *7*.
2. The set of letters in the word "mississippi".
3. The set of major league baseball players who have batting averages of over *.600*.
4. The set of days of the week having names that begin with the letter "*S*".
5. The set of even numbers between *2* and *1000*.
6. The set of odd numbers.

Say whether the statement is true or false.

7. $6 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
8. $\{5\} \in \{5, 10, 15, 20, \ldots\}$
9. $10 \in \{2, 5, 8, 11, 13, 15, 20\}$
10. $22 \in \{x : x \text{ is an even number and less than } 18\}$
11. $\{1, 2, 3, 4, 5\} = \{0, 1, 2, 3, 4, 5\}$
12. $\{1\} \in \{1, \{1\}, 2, \{2\}, 3\}$

Let $\mathcal{A} = \{2, 4, 6, 8, 10, 12\}$
$\mathcal{B} = \{1, 3, 5, 7, 9, 11\}$
$\mathcal{C} = \{4, 6\}$
$\mathcal{D} = \{3, 5\}$
$\mathcal{E} = \{4, 5\}$
$\mathcal{F} = \{x : x \text{ is an odd number}\}$
$\mathcal{G} = \{x : x \text{ is an even number}\}$

Say whether the statement is true or false.

13. $\mathcal{A} \cap \mathcal{B} = \emptyset$
14. $\mathcal{C} \cap \mathcal{D} = \mathcal{E}$
15. $\mathcal{F} \cup \mathcal{G} = \mathcal{Z}^+$
16. $\mathcal{A} \cup \mathcal{C} = \mathcal{A}$
17. $\mathcal{A} \cap \mathcal{C} = \mathcal{C}$
18. $\mathcal{A} \subseteq \mathcal{A}$
19. $3 \subseteq \mathcal{B}$
20. $\mathcal{A} \subseteq \mathcal{F}$
21. $n\mathcal{A} = 4$
22. $n\emptyset = 1$
23. $\emptyset \subseteq \mathcal{D}$
24. $\emptyset \cap \mathcal{A} = \emptyset$

Write the following sets using set builder notation

25. $\{2, 4, 6, 8, \ldots, 1000\}$
26. $\{1, 3, 5, \ldots\}$

Questions 27, 28, 29, and 29, please ***explain*** *why your answers are correct.*

27. Suppose $\mathcal{A}$ and $\mathcal{B}$ are equivalent sets. Do $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements?

28. Suppose $\mathcal{A}$ and $\mathcal{B}$ are equal sets. Are $\mathcal{A}$ and $\mathcal{B}$ equivalent sets?

29. Does the set of even integers have the same number of elements as the set of integers?

30. Is it true that: if $x \in \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{B}$, then $x \in \mathcal{B}$?